جامعة الأمير سطام بن عبدالعزيز
PRINCE SATTAM BIN ABDULAZIZ UNIVERSITY

Discrete Mathematics

جامعه الأمير سطام

د/ حاتم علي

Discrete Mathematics

CS1112

Lecture 1

Sets

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Sets

- Definitions
- Types of sets
- Cartesian product
- Finite and infinite sets
- Countable and uncountable
- Venn diagram
- Sets operations
- Laws of sets

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Sets

- A set is a collection of objects
  - Unordered
  - Objects are called elements or members of the set
  - The set contains its members
  - $x \in S$ denotes x is a member of the set S, or x belongs to the set S.
  - $x \notin S$ denotes x is not a member of the set S, or x does not belong to the set S.

Uppercase letters used to denote sets where lowercase letters are used to denote members of the set.

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Set description

- A set can be described using one of two forms
  - enumerated form (i.e. as a list)

  $A = \{2, 4, 6, 8\}$

  B = {Sun, Mon, Tue, Wed, Thur}

  - predicate form (i.e. using a property that defines the elements of the set)

  A = {x | x is an even positive integer less than 10}

  $A = \{x \mid x \in \mathbb{N}, x \text{ is even}, x < 10\}$

  B = {d | d is a working weekday}

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Some standard numerical sets

- $\mathbb{N}$ = the set of Natural numbers, the counting numbers, non-negative integers
- $\mathbb{Z}$ = the set of all integers
- $\mathbb{Z}^+$ = set of all positive integers
- $\mathbb{Z}^-$ = set of all negative integers
- $\mathbb{R}$ = the set of Real numbers
- $\mathbb{C}$ = the set of complex numbers

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Some standard numerical sets

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**

$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**

$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**

$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**

$\mathbf{R}$, the set of **real numbers**

$\mathbf{R}^+$, the set of **positive real numbers**

$\mathbf{C}$, the set of **complex numbers**.

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Empty set

- A special set with no elements
- Denoted $\varnothing$
- $\varnothing = \{\ \}$
- $\varnothing$ is a subset of every set
- Example

$X = \{x \mid x \in N \text{ and } x < 0\}$ clearly the predicate

"$x \in N$ and $x < 0$" is false, therefore $X = \varnothing$

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Equal Sets

- Two sets are said to be equal if they contain the same elements
- $A = B$ ➔ for every $x \in A$, $x \in B$ and for every $x \in B$, $x \in A$
- Example

  $A = \{a, b, c, d\}$, $B = \{c, a, b, d\}$, $C = \{b, c, b, a, a, c, d, b, a, d\}$

  $A = B = C$

  Note: repeating an element means one occurrence of that element

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The Universal Set

- The universal set, denoted by U, contains all elements that could be under discussion in a particular situation
- U changes according to circumstances
- e.g. If we are dealing with months of the year, U = {January, February, March, …, December} If we are dealing with numbers, U might be $\mathbb{R}$ (the set of all real numbers)

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Cardinality

- Cardinality of a set A, denoted | A | is the number of elements contained in the set A.
- Example

Let A = {a, ,b, c, d, e, f}, B = {a, b, c, …, z}

C = {1, 2, 3, …, 10}, D = {x | x is a student registered for Discrete Mathematics}

| A | = 6, | B | = 26, | C | = 10

| N | = ? | R | = ? | Z | = ? | D | = ? | ∅ | = ?

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Venn Diagram

- Venn diagram can be used to represent sets graphically
- The universal set U is represented by a rectangle
- Other sets – inside U – are represented by circles
- Venn diagram is usually used to represent relationships between sets
  - Can represent relationships of up to 3 sets

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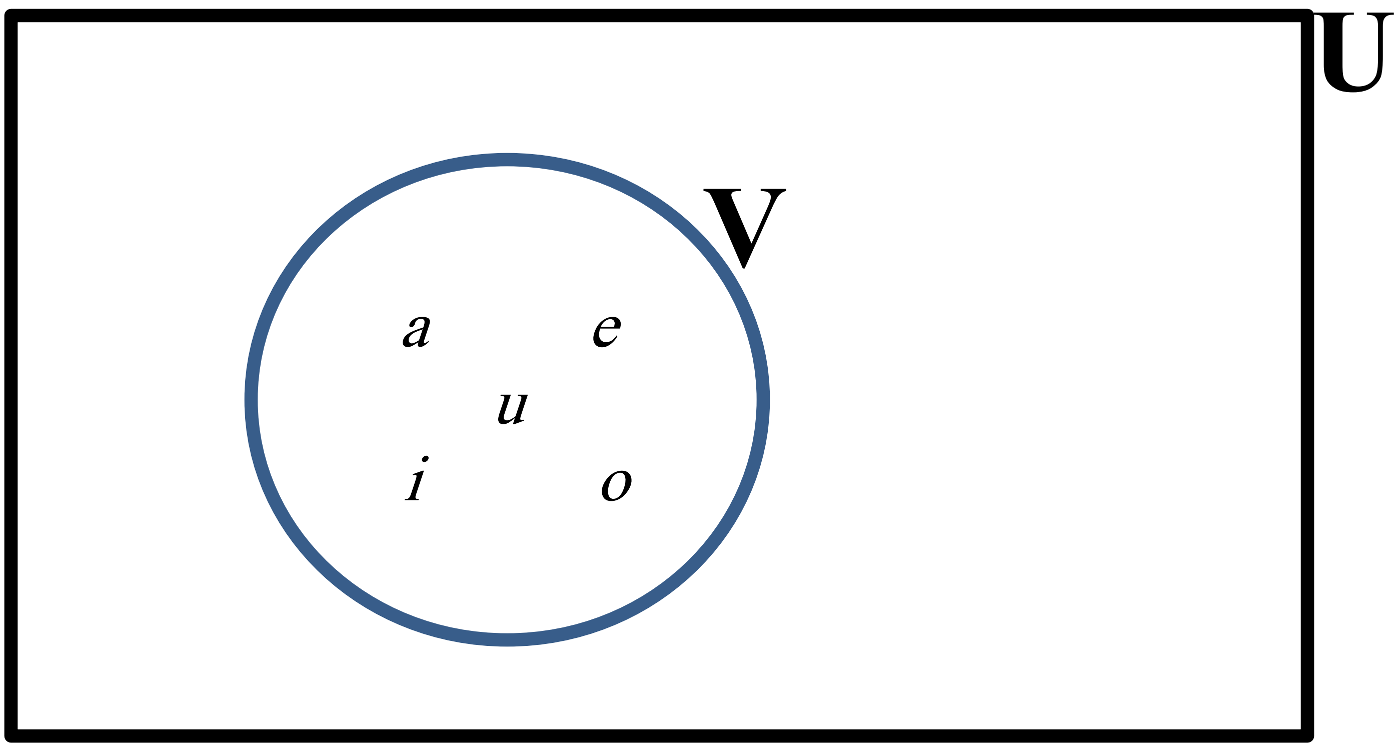
Venn Diagram



U = set of English letters
V = set of vowels

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Subsets

- The set A is a subset of the set B, denoted $A \subseteq B$, if and only if every element of A is also an element of B
  - B is a superset of A denoted $B \supseteq A$
- Example

Let A = {a, ,b, c, d, e, f}, B = {a, b, c, …, z}

C = {1, 2, 3, …, 10}

Clearly $A \subseteq B$ where A is not a subset of C

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Subsets

- Exercise
  - Describe the relationship between $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ using subset notation and Venn diagram
- Solution
  - $\mathbb{Z}^+ \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{C}$
  - $\mathbb{Z}^- \subseteq \mathbb{Z}$

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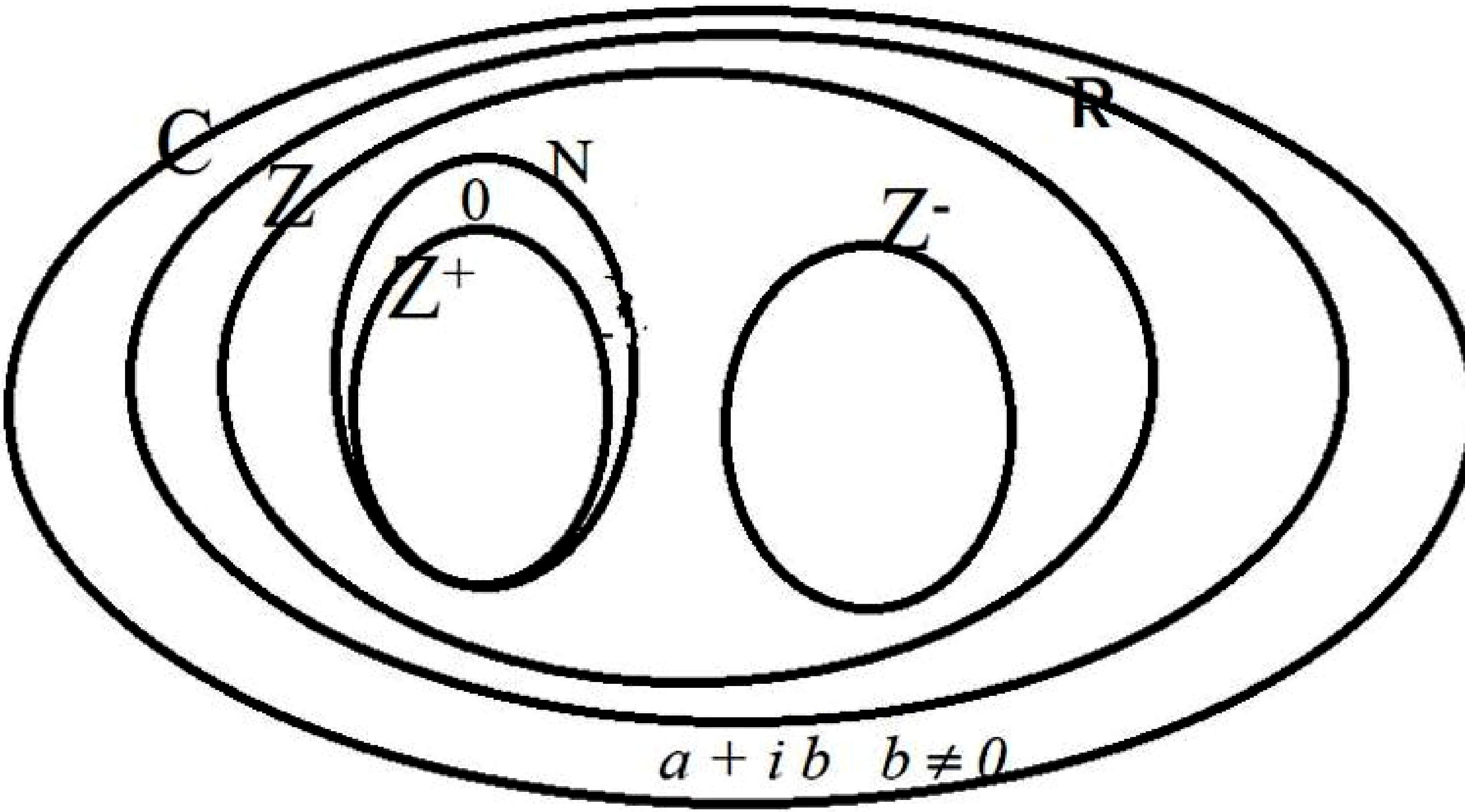
Subsets



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Subsets

- For every set X
  - $\emptyset \subseteq X$.
  - $X \subseteq X$
- For two sets A and B

  $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
- If $A \subseteq B$ and $A \neq B$ that there is at least one element in B which is not in A, then A is said to be a **proper** subset of B denoted $A \subset B$

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Power set

- Let X be a set, the set of all subsets of X, denoted P(X) is called the power set of X.
- $| P(X) | = 2^{| X |}$
- Example
  - Let A = {a, b, c}, then P(A) = { { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }
- $| P(A) | = 2^{| A |} = 2^3 = 8$

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Power set

- $P(\{1, 2\}) =$
- $P(\{h\}) =$
- The empty set and the set itself are members of the power set (the set of subsets)
- $P(\varnothing) = \{\varnothing, \varnothing\} = \{\varnothing\}$
- $P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\} = \{ \{ \}, \{ \{ \} \} \}$

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Cartesian Product

- The Cartesian product of two sets A and B denoted $A \times B$, is the set of ALL ordered pairs (a, b) such that $a \in A$ and $b \in B$
- $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- Note:

(a, b) is an ORDERED pair ➔ $(a, b) \neq (b, a)$ unless $a = b$

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Cartesian Product

- Example

Let $X = \{a, b\}$, $Y = \{1, 2, 3\}$

$X \times Y = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Note :

$| A \times B | = |A| \times |B|$

$A \times B \neq B \times A$ unless $A = \varnothing$ or $B = \varnothing$

if $A = \varnothing$ or $B = \varnothing$ then $A \times B = \varnothing$

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Cartesian Product

- $A \times B \times C = \{(a, b, c) \mid a \in A, b \in B \text{ and } c \in C\}$

- (a, b, c) is called an ordered tuple
- $(A \times B) \times C \neq A \times B \times C$

- $A^n = \{(a_1, a_2, a_3, \ldots, a_n) \mid a_i \in A, i = 1, 2, 3, \ldots, n\}$

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Set Operations

- The intersection of two sets A and B denoted $A \cap B$ is a new set containing all common elements of A and B

Elements those are members of A and members of B at the same time

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- If $A \cap B = \varnothing$, then A and B are said to be disjoint sets.

No common elements

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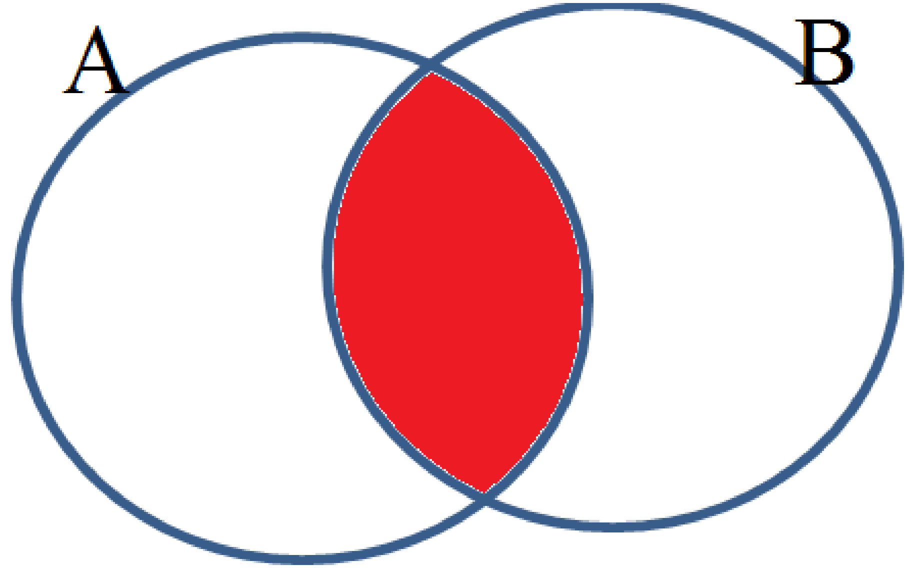
Intersection

- $A \cap B$



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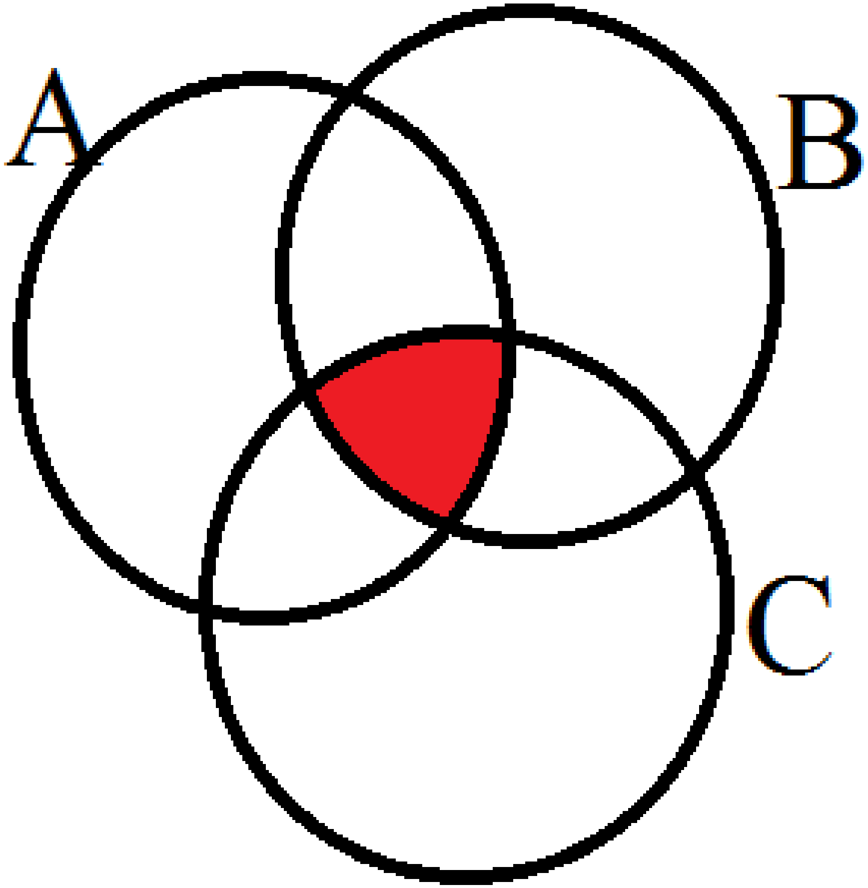
Intersection

- $A \cap B \cap C$



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- The union of two sets A and B denoted $A \cup B$ is a new set that contains all elements of A with all elements of B. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$,

- The complement of a set A denoted A is a new set consists of all the elements of the universal set that are not in A. $A = \{x \mid x \in U \text{ and } x \notin A\}$

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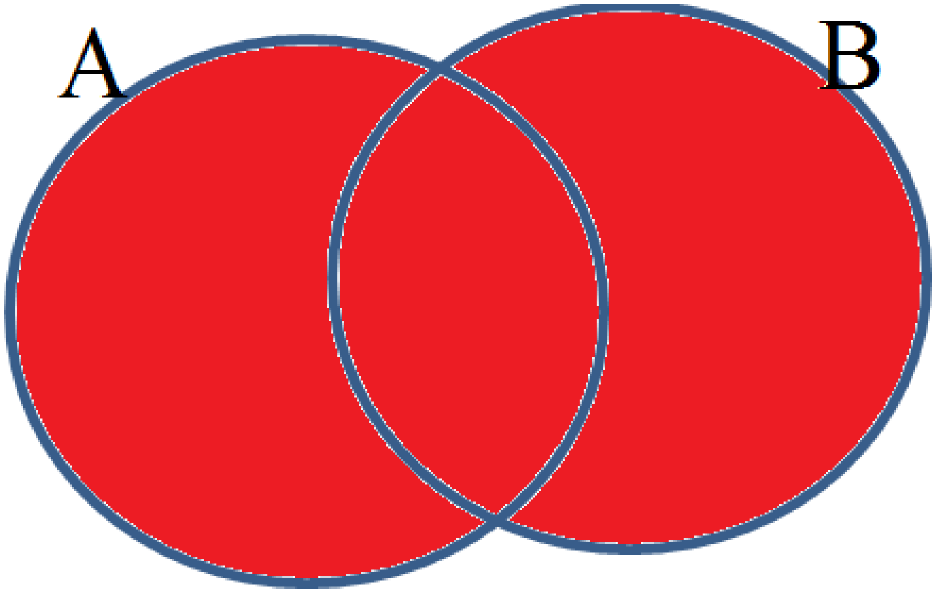
Union

- A ∪ B



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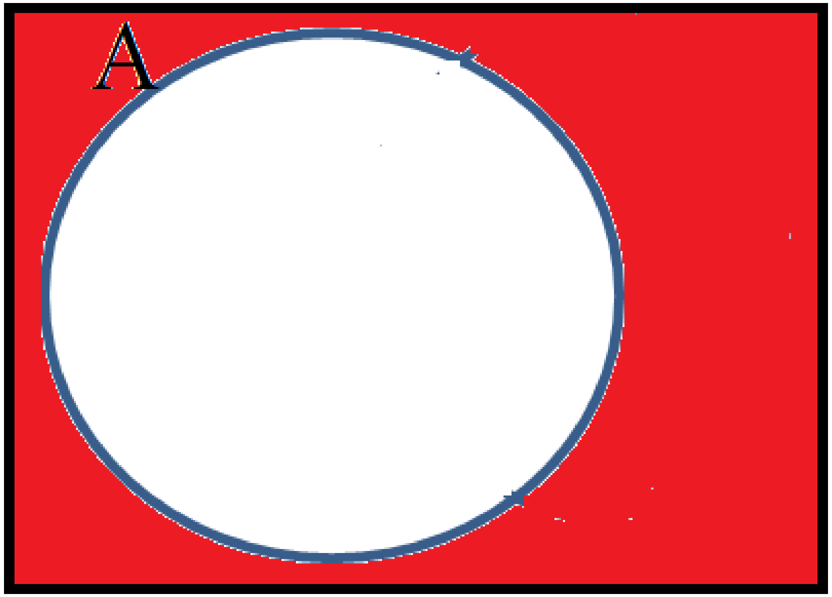
Complement

- $\overline{A}$



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- The difference of A and B, or the complement of B relative to A, denoted A - B, is the set of all elements of A which are not members of B

$$A - B = \{ x \mid x \in A \text{ and } x \notin B\}$$

- Example

$$\mathbb{Z} - \mathbb{Z}^{+} = \mathbb{Z}^{-} \cup \{ 0 \}$$

$$\mathbb{N} - \{10, 11, 12, \ldots\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$U - A = \bar{A}$$

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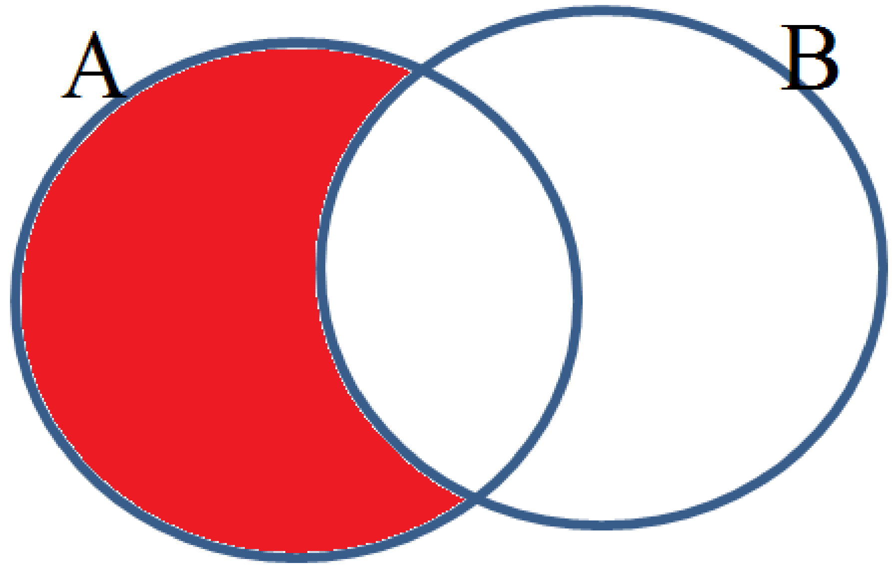
Difference

- $A - B$



- Note

  $A - B = A \cap \overline{B}$

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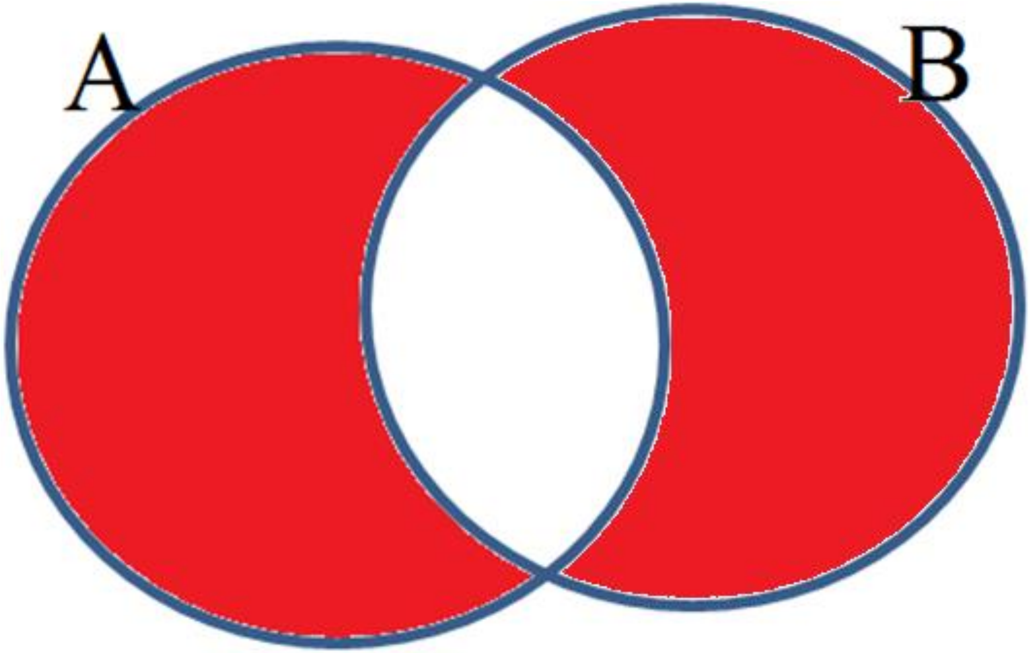
Symmetric Difference

- $A \oplus B$ or $A \Delta B$



- Note $A \oplus B = (A - B) \cup (B - A)$

$= (A \cup B) - (A \cap B)$

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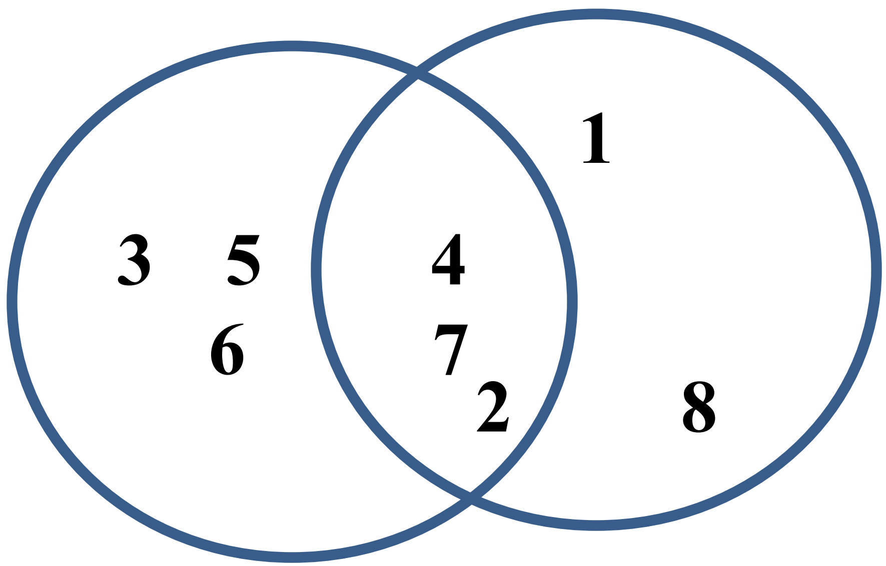
- Let A = {2, 3, 4, 5, 6, 7} B = {1, 2, 4, 7, 8}
- $A \cap B =$
- $A \cup B =$
- $A - B =$
- $B - A =$
- $A \oplus B =$



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Exercise

- Let $U = \{a, b, c, d, e, f, g, h, i, j\}$,

  $A = \{a, b, c, d, e, f, g\}$, $B = \{b, d, f, i, j\}$,

  $C = \{a, c, f, j\}$. Find:

  $A \cup C =$

  $A \cap B =$

  $\overline{A} \cap C =$

  $\overline{A} =$

  $A - B =$

  $A - B =$

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Exercise

- $A = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, $B = \{X \mid X \subseteq \{a, b\}\}$

  Is $A = B$?

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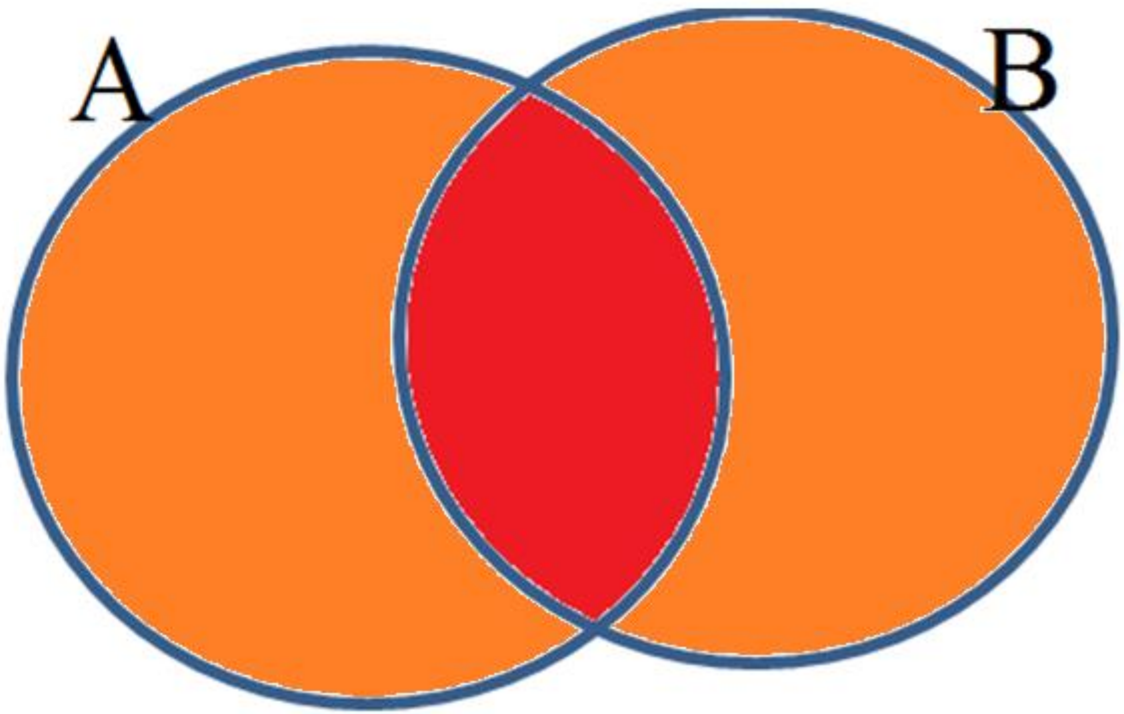
Set Operations

- $|A \cup B| = |A| + |B| - |A \cap B|$



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Laws of Sets

Law	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity Laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\overline{\overline{A}} = A$	Double complement

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Laws of Sets

Law	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's Laws

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Laws of Sets

Law	Name
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement Laws

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- $A \cup (A \cup B)$

  $(A \cup A) \cup B$ Assoc

  $(A) \cup B$ Idem

- $A \cap (A \cap B)$

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Generalized Union and intersection

For n sets $A_1, A_2, A_3, \ldots, A_n$

$$\bigcup_{i=1}^{i=n} A_i = A_1 \bigcup A_2 \bigcup A_3 \bigcup \cdots \bigcup A_n$$

$$\bigcap_{i=1}^{i=n} A_i = A_1 \bigcap A_2 \bigcap A_3 \bigcap \cdots \bigcap A_n$$

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Generalized Union and intersection

- $A_i = \{i, i+1, i+2, \ldots\}$

$$\bigcup_{i=1}^{i=n} A_i = \bigcup_{i=1}^{i=n} \{i, i+1, i+2, \ldots\} = \{1,2,3, \ldots\}$$

$$\bigcap_{i=1}^{i=n} A_i = \bigcap_{i=1}^{i=n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\}$$

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Finite & Infinite Sets

- A set S is said to be finite if there are n distinct elements
- A set is said to be infinite if it is not finite
- Example

$A = \{x \mid x \in \mathbb{N} \text{ and } x < 10\}$, $B = \{a, b, c, \ldots ,z\}$

$O = \{1, 3, 5, \ldots\}$, $E = \{2, 4, 6, \ldots\}$

$|A| = ?$, $|B| = ?$, $|O| = ?$, $|E| = ?$

A and B are finite set, where O and E are inifinite

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Countable & Uncountable Sets

- A finite set is countable
- If an infinite set has the same cardinality as the set of natural numbers it is said to be countable
- Consider the set of odd positive integers

1	2	3	4	5	6	7	8	9	10	11	12	…
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
1	3	5	7	9	11	13	15	17	19	21	23	…

Since it has a one-one correspondence with N, it is considered to be countable

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Countable & Uncountable Sets

- Exercise
- Which of the following sets is countable and which is uncountable:

$A = \{x \mid x = 2n, n \in \mathbb{N}\}$

$B = \{x \mid x = 2n + 1, n \in \mathbb{N}\}$

$C = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$

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Exercises

- Let A = {x | x is a small letter in the English alphabet} B = { y | y is a capital letter in the English alphabet }. Find
  - $|P(A)| =$
  - $|P(A \cup B)| =$
  - $|P(A \cap B)| =$
  - $|P(A \times B)| =$

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Exercises

- Let A = {x | x is a small letter in the English alphabet} B = { y | y is a capital letter in the English alphabet }. Find
  - $|P(A)| = 2^{26} = 67{,}108{,}864$
  - $|P(A \cup B)| = 2^{52} = 4.5036 \times 10^{15}$
  - $|P(A \cap B)| = |P(\emptyset)| = 2^{|\emptyset|} = 2^{0} = 1$
  - $|P(A \times B)| = 2^{|A \times B|} = 2^{|A| \times |B|} = 2^{26 \times 26} = 2^{676} = 3.1353 \times 10^{203}$

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Exercises

- There are 200 students, all of them sat for both “programming” and “discrete mathematics” exams. 150 students passed the programming exam, 150 students passed the discrete mathematics exam. No student failed both exams How many students passed both “programming” and “discrete mathematics” exams?

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Exercises

- $|\text{Passed Programming}| = 150$
- $|\text{Passed Discrete Math}| = 150$
- $|\text{Passed Programming} \cup \text{Passed Discrete Math}| = 200$
- $|\text{Passed Programming} \cup \text{Passed Discrete Math}| = |\text{Passed Programming}| + |\text{Passed Discrete Math}| - |\text{Passed Programming} \cap \text{Passed Discrete Math}|$
- $200 = 150 + 150 - |\text{Passed Programming} \cap \text{Passed Discrete Math}|$
- $|\text{Passed Programming} \cap \text{Passed Discrete Math}| = 150 + 150 - 200 = \underline{\underline{100}}$

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Sequences

- A sequence is an ordered list of elements
  - 1, 5, 9, 13 is a finite sequence
  - 1, 3, 5, …, $2n - 1$, … is an infinite sequence

Or

- A sequence is a function from a subset of integers, usually $\mathbb{Z}$ or $\mathbb{N}$, to a set S

  a_n is used to denote the image of n

  a_n is called the n^{th} term of the sequence

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Example

- Consider the sequence $\{ a_n \}$ with $$a_n = \frac{1}{n}$$

The sequence is $a_1, a_2, a_3, a_4, \ldots$

That is $$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- Consider the sequence $\{ a_n \}$ with $a_n = n^2$

The sequence is $a_1, a_2, a_3, a_4, \ldots$

That is 1, 4, 9, 16, ….

The first one is decreasing while the second is increasing

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- A sequence s_n is said to be increasing if for all n $s_n < s_{n+1}$.
- A sequence s_n is said to be decreasing if for all n $s_n > s_{n+1}$.

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Geometric Progression

- A geometric progression is a sequence of the form $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$
  - The initial term a and the common ratio r are real numbers

Example: let $a = 3$ and $r = 2$ then

the geometric progression is 3, 6, 12, 24, 48, …

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Subsequence

- Let $\{ s_n \}$ be a sequence, then the sequence $\{ t_n \}$ is said to be a subsequence of $\{ s_n \}$ if $\{ t_n \}$ contains some certain elements of $\{ s_n \}$.
- Example let $S = \{ 2n + 1 \mid n = 1, 2, 3, \dots \}$

and $T = \{ 2n^2 + 1 \mid n = 1, 2, 3, \dots \}$

Clearly $S = 3, 5, 7, \dots$ and $T = 3, 9, 19, \dots$ S contains odd integers > 1, and T contains some of the odd numbers, therefore T is a subsequence of S

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Summations

- Let $\{s_n\}$ be a sequence, then the sum

$$\sum_{i=1}^{m} s_i = s_1 + s_2 + s_3 + \ldots + s_m$$

this is called the sigma notation

The Greek letter Σ (sigma) denotes the summation of m terms from the sequence $\{s_n\}$.

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Products

- Let $\{s_n\}$ be a sequence, then the sum

$$\prod_{i=1}^{m} s_i = s_1 \times s_2 \times s_3 \times \ldots \times s_m$$

this is called the pi notation

The Greek letter Π (pi) denotes the product of m terms from the sequence $\{s_n\}$.

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Strings

- Let X be a finite nonempty set, a string α over X is a finite sequence of elements from X
- The length of a string α is the number of elements of α, denoted $|\alpha|$
- The empty/null string λ (lambda) is a string of length zero.
- Example:

  Let X = {a, b, c, d} and α = bbcaaca, then $|\alpha| = 7$

  $\alpha = bbcaaca = b^2ca^2ca$

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Strings

- X^* = {set of all strings over X including λ}
- $X^+ = X^* - \{ \lambda \}$, the set of all nonempty strings
- Example
  - Let A = {A, B, C, …, Z, a, b, c, …, z}
  - A^* is the set of all possible strings/words including the null string
  - A^+ is the set of all possible strings/words excluding the null string
  - The set of all English words is a subset of A^+.

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Strings

- Concatenation of two strings is a new string formed by putting one string after the other, or writing one string *followed by* the other string
- Example

  Let α = my and β = computer then $\alpha\beta$ = mycomputer and $\beta\alpha$ = computermy

  $| \alpha\beta | = | \beta\alpha | = | \alpha | + | \beta |$

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